WORKSHOP: PROFINITE RIGIDITY AT KIT

JONATHAN FRUCHTER

Contents

Introduction	1
1. Overview: classics, important theorems and open questions	2
2. Remeslennikov's question	5
3. Profinite trees and Bass-Serre theory	8
References	11

Introduction

To set the stage, we begin by fixing relevant terminology; throughout these notes all groups are assumed to be finitely generated and residually finite (that is, every group element survives in a finite quotient). Under this assumption, a group G embeds in its profinite completion $\hat{G}$, and $\hat{G}$ is a *totally disconnected, compact and Hausdorff* topological group. We use $\mathcal{C}$ as a generic notation for a class of groups (e.g. the class of all hyperbolic, finite-volume 3-manifold groups) of finitely generated, residually finite groups.

Definition 0.1. Let G be a finitely generated, residually finite group. The *(profinite) genus* of G is the collection

$$\mathfrak{g}(G) = \{H \mid H \text{ is finitely generated, residually finite and } \hat{H} \cong \hat{G}\}.$$

Similarly, the $\mathcal{C}$*-genus* of G is given by

$$\mathfrak{g}_{\mathcal{C}}(G) = \{H \in \mathcal{C} \mid H \text{ is finitely generated, residually finite and } \hat{H} \cong \hat{G}\}.$$

We say that G is

(1) *(absolutely) profinitely rigid*, if $\mathfrak{g}(G) = \{G\}$;
(2) *almost profinitely rigid*, if $\#\mathfrak{g}(G) < \infty$;
(3) *profinitely rigid within (alternatively among, in etc.,)* $\mathcal{C}$ if $\mathfrak{g}_{\mathcal{C}}(G) = \{G\}$;
(4) *almost profinitely rigid within* $\mathcal{C}$ if $\#\mathfrak{g}_{\mathcal{C}}(G) < \infty$.

Another useful notion is the following:

Definition 0.2. A property $\mathcal{P}$ is a *profinite invariant* (or, a *profinite property*) if for every finitely generated, residually finite group G and every $H \in \mathfrak{g}(G)$, G has $\mathcal{P}$ if and only if H does. We say that $\mathcal{P}$ is a profinite invariant *within the class* $\mathcal{C}$ if for every $G \in \mathcal{C}$ and $H \in \mathfrak{g}_{\mathcal{C}}(G)$, G has $\mathcal{P}$ if and only if H has $\mathcal{P}$.

The biggest question in the field of profinite rigidity is the following question of Remeslennikov:

Question 0.3. *Let G be a finitely generated, residually finite group such that $\hat{G}$ is a non-abelian free profinite group. Does it follow that G is a free group?*

Many of the results we mention and prove in the series are closely intertwined with this question; as a matter of fact, the second talk will be devoted entirely to this question. The first talk that we give will be a general overview of the field of profinite rigidity. Lastly, the third talk will focus on a specific aspect of the study of profinite groups: profinite Bass-Serre theory.

1. Overview: classics, important theorems and open questions

An "early form" of profinite rigidity is the following lemma (often referred to as *Dixon's theorem*); in fact, it has become so standard that people commonly say that G is profinitely rigid if there is no other finitely generated, residually finite group with the same set of (isomorphism types) finite quotients as that of G. We fix the following notation:

Notation 1.1. For a group G, denote by $\mathscr{F}(G)$ the set of (isomorphism types) of finite quotients of G.

Lemma 1.2 ([DFPR82, Main Theorem])**.** *Let G and H be finitely generated, residually finite groups. Then $\hat{G} \cong \hat{H}$ if and only if $\mathscr{F}(G) = \mathscr{F}(H)$.*

Proof. The implication $\Longrightarrow$ is clear; suppose if so that $\mathscr{F}(G) = \mathscr{F}(\hat{G}) = \mathscr{F}(\hat{H}) = \mathscr{F}(H)$. Let $\hat{G}_n$ be the intersection of all finite-index normal subgroups of $\hat{G}$ of index at most n, denote $G \geq G_n = \hat{G}_n \cap G$ and define $\hat{H}_n$ and H_n similarly. Note that G_n (respectively, $\hat{G}_n$) is normal in G (respectively, $\hat{G}$); in fact, G_n is a characteristic subgroup of G. In addition, since G is finitely generated, for every k there are finitely many subgroups of G of index k (as these correspond to homomorphisms $G \to S_k$). It follows that G_n is the intersection of finitely many finite-index subgroups of G, and $[G : G_n] < \infty$. Lastly, note that the inverse system given by the G_n's is cofinal, meaning that $\hat{G} = \varprojlim G/G_n$.

We next observe that G/G_n can be characterized as the (unique) largest finite quotient Q of G satisfying the following property: the intersection of all normal subgroups of Q of index $\leq n$ is trivial. Indeed, the intersection of the preimages of all such subgroups is exactly G_n, so G_n is killed in every such quotient Q. Since $\mathscr{F}(G) = \mathscr{F}(H)$, all that remains to do is construct an *isomorphism of inverse systems* between $(G, \{G_n\})$ and $(H, \{H_n\})$. This amounts to finding maps $f_n : G/G_n \to H/H_n$ that preserve the structure of the inverse systems.

We construct such maps implicitly, by defining a new inverse system and taking its limit. For every n, let A_n be the collection of all isomorphisms $G/n \to H/n$. Let $g_n \in A_n$ and $m \leq n$; since g_n is an isomorphism, it must map the image of G_m in G/G_n (that is, G_m/G_n) to H_m/H_n. Therefore g_n gives rise to a quotient map

$$h_{n,m} : (G/G_n)/(G_m/G_n) = G/G_m \to (H/H_n)/(H_m/H_n) = H/H_m$$

such that the collection $(\{A_n\}, \{h_{n,m}\})$ is an inverse system. Any element (f_n) in its inverse limit gives the desired isomorphism $f = (f_n) : \varprojlim G/G_n \to \varprojlim H/H_n$. $\square$

1.1. Basic profinite properties, examples and open questions.

In Lukas Schneider's talk we saw that finitely generated abelian groups are (absolutely) profinitely rigid, that the abelianization of a group is a profinite invariant, and that being profinitely rigid is not preserved by going to finite-index overgroups (or subgroups). We highlight the following more general phenomenon:

Remark 1.3. Suppose that a group G satisfies a law (that is, there exists a word w in a free group F, such that for every tuple of elements $\overline{g}$ from G, $w(\overline{g}) = 1$). Then every $H \in \mathfrak{g}(G)$ satisfies the same law.

Example 1.4. One example of a law is being *nilpotent of height* n. It follows that being nilpotent (and more specifically, nilpotent of a certain height) is a profinite invariant.

Recall that there is a correspondence between the finite-index subgroups of G and $\hat{G}$, given by (draw diag)

$$\begin{array}{ccc} H \leq G & \mapsto & \overline{H} \leq \hat{G} \\ F \leq \hat{G} & \mapsto & F \cap G. \end{array}$$

This implies that the collection of abelianizations, its free and torsion parts, and their "filtered" counterparts (e.g. the collection of pairs (T, n) where T is the torsion part of H^{ab} and H is a subgroup of G of index n) are profinite invariants.

Remark 1.5. Looking at the (torsion part of the) abelianization of finite-index subgroups is the most common computational methods used to distinguish groups profinitely, see for example the supplemental code to [BMRS20], as well as [Gar19]. In a forthcoming paper, Ascari and the author prove that finitely generated groups acting freely on $\mathbb{R}$-trees are profinitely rigid within the class of residually free groups, by showing that every limit group that does not admit a free action on an $\mathbb{R}$-tree has a finite-index subgroup with torsion in its abelianization.

This also suggests that (when Lück's approximation Theorem applies) the first ℓ^2-Betti number is a profinite invariant, as we will see in Sruthy Joseph's talk. In general, "most" properties are *not* profinite invaraiants, and the examples often come from the realm of arithmetic groups. Some non-invariant properties are:

(1) Amenability [KS23],
(2) Property (T) [Aka12],
(3) Property `FA` [CWLRS22] (and more generally, [Bri23] for the higher-dimensional generalization of fixed points for actions on d-dimensional CAT(0) spaces),
(4) Having vanishing second bounded (real) cohomology [EK23]

We conclude the discussion with two (important and difficult) open questions:

Question 1.6. *Is the rank $d(G)$ of a group G (that is, the minimal size of a generating set) a profinite invariant?*

Contrary to property `FA` *not* being a profinite property, the following is still open:

Question 1.7. *Is one-endedness a profinite invariant?*

1.2. **Key results.** One of the earliest profinite rigidity results is the following generalization of the fact that finitely generated abelian groups are profinitely rigid:

Theorem 1.8 ([Pic71, Main Theorem])**.** *Let G be a finitely generated nilpotent group; then G is almost profinitely rigid.*

Remark 1.9. Note that every finitely generated nilpotent group is residually finite by a classical theorem of Hirsch's. In addition,

(1) As already discussed, nilpotency is a profinite invariant; therefore, Pickel's result is originally stated only for finitely generated nilpotent groups.
(2) Pickel's proof goes through proving that if G and H are finitely generated nilpotent groups with $\hat{G} \cong \hat{H}$, then their torsion subgroups $t(G)$ and $t(H)$ are isomorphic; furthermore, $\mathscr{F}(G/t(G)) = \mathscr{F}(H/t(H))$.

Remark 1.10. Recently, a different kind of generalization of the fact that finitely generated abelian groups are profinitely rigid emerged in the form of *affine Coxeter groups*, see more in [CHMV24] and independently [PS24].

For a long time, these have been the only examples of (almost) profinitely rigid groups; a few years ago, Bridson, McReynolds, Reid and Spietler proved the following breakthrough, which were the first examples of profinitely rigid groups containing non-abelian free groups:

Theorem 1.11 ([BMRS20, Theorems 7.1 and 9.1])**.** *The group* $\mathrm{PSL}(2, \mathbb{Z}[\omega])$*, where* ω *is a third root of unity, is profinitely rigid; the fundamental group of the Weeks manifold (the closed, hyperbolic* 3*-manifold of minimal volume) is profinitely rigid.*

Remark 1.12. The above set of authors gave further examples of profinitely rigid groups, this time in the form of Fuchsian triangle groups [BMRS21].

The strategy behind the proof of these theorems, can be summarized as follows:

(1) The authors study lattices Γ in $\mathrm{PSL}(2, \mathbb{C})$ that have very few irreducible representations into $\mathrm{SL}(2, \mathbb{C})$ (up to conjugation).
(2) Then, the representations of Γ into $\mathrm{SL}(2, \mathbb{C})$ are used to construct representations $\hat{G} \to mathrmSL(2, \overline{\mathbb{Q}}_p)$ (here $\overline{\mathbb{Q}}_p$ is the algebraic closure of the p-adic rationals).
(3) Now, if Λ is a finitely generated, residually finite group with the same profinite completion as that of Γ, $\hat{\Lambda}$ admits the same representations into $mathrmSL(2, \overline{\mathbb{Q}}_p)$.
(4) Restricting these representations to Λ, the next step is to force the image of λ to live inside (a finite extension of) that of Γ.
(5) Using "3-dimensional" arguments, they show that the map $\Lambda \to \Gamma$ must be surjective. Considering the induced morphism on profinite completions, and using the *Hopf property* (that will be discussed in the next lecture) finished the proof.

Unfortunately, these result subsume almost all known examples of absolutely profinitely rigid groups; other known examples utilize mostly techniques inspired by the works above (see, for example, [CW22]).

Another significant recent breakthrough, is due to Andrei Jaikin-Zapirain; this is the *only* result that tells us something concrete about groups lying in $\mathfrak{g}(F)$, where F is a non-abelian free group:

Theorem 1.13 ([JZ23, Theorem 1.1])**.** *Let* $G \in \mathfrak{g}(F)$*, where* F *is a non-abelian free group. Then* G *is residually-*p *for every prime* p*, and* G *is* parafree *(meaning that* $G/\gamma_n(G) \cong F/\gamma_n(F)$ *for every* n*).*

Jaikin-Zapirain's proof is very algebraic in nature, and relies on utilizing a universal ring that encodes all the representations of the group, and computing its dimension in two ways.

The last result that we bring forth, is Grothendieck's *original* question about profinite completions:

Question 1.14 (Grothendieck's question, 1970)**.** *Given two finitely presented, residually finite groups* G *and* H*, and a homomorphism* $f : G \to H$ *such that* $\hat{f} : \hat{G} \to \hat{H}$ *is an isomorphism. Must* f *be an isomorphism?*

Remark 1.15. A *Grothendieck pair* is a pair of groups $H \leq G$ such that the inclusion map $i : H \to G$ induces an isomorphism of profinite completions. If G does not admit such a subgroup H, G is called *Grothendieck rigid.*

Progress towards solving Grothendieck's problem was first made by Platonov and Tavgen [PT86], where they proved that a direct product of two free groups $F_2 \times F_2$ has a finitely generated subgroup G, such that $G \leq F_2 \times F_2$ is a Grothendieck pair. We remark that there are infinitely many such subgroups of $F_{@} \times F_2$, but *none* of them is finitely presented (this follows, for example, from the fact that finitely presented subgroups of residually free groups are separable [BW07]). Bridson and Grunewald managed to overcome this problem by utilizing the *Rips Construction*, producing hyperbolic groups with a desired quotient.

Remark 1.16. It is worth mentioning that by [FM24], direct products of free, surface and free abelian groups are profinitely rigid within finitely presented, residually free groups; note that the examples of Platonov and Tavgen are all residually free, as they are subgroups of $F_2 \times F_2$.

Theorem 1.17 ([BG04, Theorem 1.1])**.** *There exist hyperbolic groups G such that $G \times G$ admits a finitely presented subgroup H, and such that the inclusion $H \hookrightarrow G \times G$ induces an isomorphism of profinite completions.*

We conclude by giving a partial proof (namely, the easy part of the proof) of Platonov's and Tavgen's result:

Lemma 1.18. *$F_2 \times F_2$ has a finitely generated subgroup H of infinite index, which is dense in the profinite topology.*

Proof. The construction of H presented below relies on the existence of a 2-generated, finitely presented, infinite simple group (as a side note, we remark that there are no known examples of d-generated finitely presented, infinite simple groups for $d > 2$). In fact, we will explicitly use Thompson's group V, which is one of the most famous examples of infinite simple groups, and which admits a finite presentation with two generators u and v and 7 relations. We remark that it is enough to use a group that admits a finite presentation on 2 generators that is not simple, but has no finite-index subgroups.

Fix an epimorphism $f : F_2 \twoheadrightarrow V$ and let H be the corresponding fibre product, that is

$$H = \{(g, h) \in F_2 \times F_2 | f(g) = f(h)\}.$$

Since V is finitely presented, H is finitely generated [BR84, Lemma 2], and H is easily seen to be of infinite index since V is infinite. Finally, to show that H is profinitely dense in $F_2 \times F_2$, it is enough to show that it is not contained in a proper finite index subgroup G of $F_2 \times F_2$. Indeed, if it were contained in such G, then $G/(\ker f \times \ker f)$ would be a proper finite index subgroup of $V \times V$, but these do not exist since V is simple. □

We conclude the discussion by mentioning that recently, Bridson, Reid and Spietler gave an example of a group G such that $G \times G$ is not profinitely rigid, but it is profinitely rigid within the class of finitely presented groups [BRS23].

2. Remeslennikov's question

The nature of this talk is more topological/geometric than the previous one; the main theme is detecting a desired topological/geometric property, and translating it into an algebraic property that can be seen by the profinite completion.

The biggest question in the field of profinite rigidity (which, for the untrained eye looks like a first-year algebra exercise), is *Remeslennikov's question*:

Question 2.1 (Remeslennikov's question)**.** *Is a finitely generated, non-abelian free group F profinitely rigid?*

As a first exercise, we will show that a free group can be profinitely distinguished from a surface group:

Example 2.2. Let F be a free group and let S be the fundamental group of a closed surface; suppose that $\hat{F} \cong \hat{S}$. Note that S has a degree-2 cover S' which is an orientable surface group, and hence $S'^{ab} \cong \mathbb{Z}^{2n}$ for some n. It follows that F has an index 2 subgroup F' with the same abelianization, and it is therefore a free group of rank $2n$. By the Nielsen-Schreier formula, the rank of F' is $1 + 2 \cdot (d(F) - 1)$ which is odd, contradicting our assumption.

Our next simple observation, is that every $G \in \mathfrak{g}(F)$ must satisfy $d(G) > d(F)$. To prove this, we will first show that profinite groups satisfy the *Hopf* property, namely:

Definition 2.3. We say that a group G is *Hopfian* if every epimorphism $G \to G$ is an isomorphism. We say that a topological group G is *(topologically) Hopfian* if every continuous epimorphism $G \to G$ is a homeomorphism.

First, we remark that non-abelian free groups have the Hopf property: if $f : F \to F$ is a surjective homeomoprhism, the image of a basis of F is a generating set of size $d(F)$ and therefore a basis, which implies that f is an isomorphism. Our next aim is to show that all topologically finitely generated profinite groups have the (topological) Hopf property:

Theorem 2.4. *Let G be a topologically finitely generated profinite group. Then G has the Hopf property.*

Proof. Let $f : G \to G$ be a surjective map; since f is surjective, for every finite-index normal subgroup $H \leq G$ we have that $[G : f^{-1}(H)] = [G : H]$. We deduce that $f(G_n) \leq G_n$ and that $G_n \leq f^{-1}(G_n)$ (for G_n as in lemma 1.2). This implies that f descends to a map $f_n : G_n \to G_n$ (the kernel of the map $G \xrightarrow{f} G \to G/G_n$ contains G_n so the map factors via the quotient $G \to G_n$). The map f_n is surjective, and since G/G_n is finite, f_n is an isomorphism. The inverse limit $\varprojlim f_n$ is an isomorphism $G \to G$, and it coincides with f. ☐

We deduce the following:

Theorem 2.5 (*cf.* [DFPR82])**.** *Let F be a non-abelian free group. Then for every $G \in \mathfrak{g}(F)$, $d(G) > d(F)$.*

Proof. We first note that if G and H are topologically finitely generated profinite groups, and $f : G \to H$ and $f' : H \to G$ are continuous surjections, then f and f' are isomorphisms. Indeed, $f' \circ f$ is an epimorphism $G \to G$, so by the Hopf property f is injective; the opposite composition shows that f' is injective too. Now both maps are continuous bijections from a compact space to a Hausdorff space, so they are homeomorphisms.

We are now ready to prove the theorem: suppose that $d(G) \leq d(F)$, so there is an epimorphism $f : F \to G$; this induces a continuous epimorphism $\hat{f} : \hat{F} \to \hat{G} \cong \hat{F}$ and by the (topological) Hopf property, $\hat{f}$ is a homeomorphism. In particular, $\hat{f}$ is injective, so f is injective and $G \cong F$. ☐

Remark 2.6. Note that the only place where we used that F is free was to obtain an epimorphism $F \to G$. It follows that whenever we have a group homomorphism $G \to H$ between two groups with the same set of finite quotients, then $G \cong H$.

It is an easy exercise to see that the above can be rephrased as follows:

Theorem 2.7. *Let G be a finitely generated, residually finite group and let F be a free group. Suppose that $\mathscr{F}(G) = \mathscr{F}(F)$, and that G has a finite quotient Q such that $d(G) = d(Q)$. Then $G \cong F$.*

2.1. **Residually free groups.** We continue with a *relative* solution of Remeslennikov's question, within the class of *residually free* groups (that is, groups in which every element survives in a *free* quotient). The proof crucially relies on the fact that residually free groups, except for free groups, always contain surface subgroups (here we count $\mathbb{Z}^2$ as a surface subgroup). In fact, relating to Jaikin-Zapirain's theorem [JZ23], people believe that a positive answer to Remeslennikov's question will go through proving that parafree groups admit surface subgroups.

Theorem 2.8 (Based on [Wil18, Corollary D])**.** *Free groups are profinitely rigid within the class of residually free groups.*

The proof relies on another theorem of Wilton's, which states that every finitely generated subgroup of *limit groups* are closed in the profinite topology in a very strong way. We recollect the relevant definitions and lemmas before we prove Theorem 2.8 above.

Definition 2.9. A *limit group* L is a finitely generated, *fully residually free* group, meaning that for every finite subset $S \subset L$ there is a homomorphism $f : L \to F$ where F is a free group, such that f is injective on S.

Definition 2.10. Let G be a group. A subgroup $H \leq G$ is called a *retract* of G if there is a homomorphism $r : G \to H$ such that $r|_H$ is the identity on H. We say that G *virtually retracts onto* H if there is a finite-index subgroup $G' \leq G$ such that H is a retract of G'.

Remark 2.11. Note that if G is residually finite, and G virtually retracts onto H, then H is separable in G (or in other words, H is closed in the profinite topology on G). Indeed, one easily checks that if G_i are finite-index subgroups of G such that $\bigcap_i G_i = \{1\}$, then $H = \bigcap_i (\ker r \cap G_i) \cdot G'$ (where $r : G' \to H$ is a retraction).

Theorem 2.12 ([Wil08, Theorem B])**.** *Let L be a limit group and let H be a finitely generated subgroup of L. Then L virtually retracts onto H.*

The following simple observation is a crucial ingredient in the proof of Theorem 2.8, and shows that virtual retracts serve as good means for transferring data from arbitrary subgroups to finite-index ones:

Lemma 2.13. *Suppose that G virtually retracts onto its subgroup H, then G has a finite-index subgroup G' such that for every coefficient module R, and any $n \in \mathbb{N}$, $H^n(H; R) \hookrightarrow H^n(G'; R)$.*

Proof. Let G' be a finite-index subgroup of G that retracts onto H via $r : G' \to H$ and let $i : H \to G'$ be the inclusion map. Note that $r \circ i : H \to H$ is the identity map. Therefore, the induced map on cohomology, $i^* \circ r^* : H^n(H; R) \to H^n(H; R)$ is the identity map. In particular, the map $i^* : H^n(H; R) \to H^n(G'; R)$ is injective. $\square$

Proof of theorem 2.8. Let G be a residually free group and suppose that $\hat{G} \cong \hat{F}$. If G is not a limit group, it is a classical result that G contains $F_2 \times \mathbb{Z}$ as a subgroup and therefore $\mathbb{Z}^2 \leq G$. Furthermore, since $\mathbb{Z}^2$ and all of its subgroups are finitely presented, by [BW07] G induces the full profinite topology on this subgroup. Hence $\overline{\mathbb{Z}}^2 = \hat{\mathbb{Z}}^2$ is a subgroup of $\hat{G}$, but a free profinite group does not contain such subgroup.

We deduce that G is a limit group, and therefore G contains a surface subgroup S. There is a virtual retraction $r : G' \to S$, and by the previous lemma $H^2(G'; \mathbb{Z}/2) \neq 0$. As we will see in Raquel Murat's talk, limit groups are *good in the sense of Serre* which means that $H^2(\hat{G}'; \mathbb{Z}/2) \neq 0$. Note that $\hat{G}'$ is a finite-index subgroup of $\hat{F}$, and is therefore a free profinite group. But this implies that $H^2(\hat{G}'; \mathbb{Z}/2) = 0$, a contradiction. □

3. Profinite trees and Bass-Serre theory

In this talk we will give a brief overview of the theory of *profinite trees*; the main theorem of Bass-Serre, namely the correspondence between group splittings and actions on trees, can be carried out in the profinite setting. However, as one might expect, some things behave less nicely in the profinite world (whereas other things behave better). We will also prove a simple fixed-point theorem for profinite *Poincaré duality* groups acting on profinite trees. We will finish by sketching the proof of two results concerning profinite completions of 3-manifold groups: profinite detection of hyperbolic geometry, and profinite completion of the Knesser-Milnor decomposition.

3.1. **Profinite graphs and trees.** Just like profinite groups are inverse limits of finite groups, one can consider *profinite spaces* which are inverse limits of finite. discrete spaces. We begin by defining *profinite graphs*. We use the following convention when considering abstract graphs: a graph Γ is a disjoint union of two sets $E \sqcup V$, with two maps $d_0, d_1 : \Gamma \to V$, such that $d_0|_V = d_1|_V = \mathrm{Id}_V$. These are interpreted in the following way: for an edge $e \in E$, we interpret $d_0(e)$ as the *initial vertex* of e, and $d_1(e)$ as the terminal vertex of e.

Definition 3.1. A *profinite graph* Γ is a graph such that

(1) Γ is a profinite space (that is, Γ is the inverse limit of its finite quotient graphs),
(2) V is a closed subset of Γ,
(3) the maps d_0 and d_1 are continuous.

Note that we didn't require that the set E is closed.

Example 3.2. Every finite graph is a profinite graph. We give another example of a profinite graph: let $V = \mathbb{N} \cup \{\infty\}$ and let $E = \mathbb{N}$, and set $d_0(n) = n$ and $d_1(n) = n+1$. Then Γ is the inverse limit of the graphs which are a finite path, with maps being collapsing the last edge of the path of length $n+1$ to obtain the path of length n. Note that there is no edge connected to the vertex ∞, yet this graph is still considered *connected*:

Definition 3.3. A profinite graph Γ is said to be *connected* if it is the inverse limit of finite, connected graphs.

This example tells us that paths have less importance in the realm of profinite graphs than in that of abstract graphs. Therefore, it would not make sense to define a profinite tree as a profinite graph without (reduced) loops. Instead, profinite trees are defined by a homological condition:

Definition 3.4. A profinite graph Γ is a *profinite tree* if it is connected, and satisfies $H_1(\Gamma, \hat{\mathbb{Z}}) = 0$ (or, equivalently, $H_1(\Gamma, \mathbb{F}_p) = 0$ for all p). In more detail, Γ is a profinite tree if and only if the following sequence is exact for all p:

$$0 \longrightarrow [\![\mathbb{F}_p[E^*, *]]\!] \xrightarrow{\delta} [\![\mathbb{F}_p[V]]\!] \xrightarrow{\epsilon 0}$$

where:

(1) $(E^*, *)$ is the quotient Γ/V,
(2) $[\![\mathbb{F}_p[E^*, *]]\!]$ is the inverse limit of the pointed spaces $\mathbb{F}_p[E(Gamma_i), *]$ (that is, all maps in the system send the base point to the base point),
(3) $[\![\mathbb{F}_p[V]]\!]$ is the inverse limit of $\mathbb{F}_p[V(\Gamma_i)]$,
(4) the map ϵ sends the image of every $v \in V$ in $[\![\mathbb{F}_p[V]]\!]$ to 1, and
(5) the map δ sends the image of every $e \in E$ in $[\![\mathbb{F}_p[E^*, *]]\!]$ to $d_1(e) - d_0(e)$.

Remark 3.5. One can now define:

(1) $H_0(\Gamma, \mathbb{F}_p) = \ker \epsilon / \mathrm{im}\delta$. Note that $H_0(\Gamma, \mathbb{F}_p) = 0$ for all p if and only if Γ is connected.
(2) $H_1(\Gamma, \mathbb{F}_p) = \ker \delta$.

One easily verifies that a finite tree is a profinite tree; moreover, the inverse limit of finite trees is always a profinite tree.

3.2. **Profinite graphs of groups.** *Profinite graphs of groups* are simply graphs of groups where the vertex and edge groups are profinite groups, and then edge monomorphisms are continuous maps. To define the *profinite fundamental group* of a profinite graph of groups, we take two approaches: one, as a completion (not always the profinite completion) of the abstract fundamental group, and one by a universal property.

Definition 3.6. Let $\mathcal{G}$ be a profinite graph of groups. The *profinite* fundamental group of $\mathcal{G}$, $\prod_1(\mathcal{G})$, is defined by any of the following two equivalent ways:

(1) Let $\mathcal{N} = \{N_i | i \in I\}$ be the collection of all normal subgroups of the abstract fundamental group G of $\mathcal{G}$, such that for every vertex gruop $\mathcal{G}_v$ of $\mathcal{G}$, $N \cap \mathcal{G}_v$ is open in $\mathcal{G}_v$. Then

$$\prod_1(\mathcal{G}) = \varprojlim_{N \in \mathcal{N}} G/N.$$

(2) $\prod_1(\mathcal{G})$ is the profinite group satisfying the same universal property as in the abstract case, except that here we request all morphisms to be continuous.

Remark 3.7. Unlike the abstract case, in the profinite setting the vertex and edge groups do not necessarily embed in $\prod_1(\mathcal{G})$. We will be especially interested in the profinite completion of graphs of groups. Suppose that $\mathcal{G}$ is a graph of abstract groups and denote by $\hat{\mathcal{G}}$ the graph of groups obtained by taking the profinite completion of the vertex and edge groups. Suppose that $\mathcal{G}$ is efficient, that is

(1) $\pi_1(\mathcal{G})$ is residually finite,
(2) each edge and vertex group is closed in the profinite topology on $\pi_1(\mathcal{G})$, and
(3) $\pi_1(\mathcal{G})$ induces the full profinite topology on the edge and vertex groups.

Note that these conditions always hold when $\pi_1(\mathcal{G})$ is subgroup separable. In this case, the edge groups of $\hat{\mathcal{G}}$ do embed in the vertex groups, and moreover

$$\widehat{\pi_1(\mathcal{G})} \cong \prod_1(\hat{\mathcal{G}}).$$

3.3. **Hierarchies and 3-manifold geometries.** We seal the discussion by sketching the proof of the following theorem, due to Wilton and Zalesskii:

Theorem 3.8 (*cf.* [WZ17, Theorem A])**.** *Let M and N be two closed, orientable, aspherical 3-manifolds with $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$. Then M is hyperbolic if and only if N is hyperbolic.*

Our strategy will be to show that M is hyperbolic if and only if $\widehat{\pi_1(M)}$ does not contain a copy of $\hat{\mathbb{Z}}^2$. The fact that abelian subgroups of 3-manifold groups are separable [Ham01, Theorem 1] reduces the question to showing that the profinite completion of the fundamental group of a closed, orientable, aspherical and hyperbolic 3-manifold does not contain $\hat{\mathbb{Z}}^2$:

Lemma 3.9. *Let M be the fundamental group of a non-hyperbolic, closed, orientable and aspherical 3-manifold. Then $\widehat{\pi_1(M)}$ contains a copy of $\hat{\mathbb{Z}}^2$.*

Proof. By Thurston's hyperbolization theorem, $\pi_1 M$ contains $H \cong \mathbb{Z}^2$ as a subgroup. Every finite subgroup of H is isomorphic to H, and therefore separable in $\pi_1 M$. It follows that $\pi_1 M$ induces the full profinite topology on H, and therefore $\overline{H} \leq \widehat{\pi_1(M)}$ is isomorphic to $\hat{\mathbb{Z}}^2$. □

Recall that by Agol's and Wise's work, fundamental groups of closed, hyperbolic 3-manifolds are virtually special. This implies that in order to prove Theorem 3.8 above, it is enough to prove the following:

Theorem 3.10. *Let G be a hyperbolic, torsion-free, virtually compact special group. Then $\hat{G}$ does not contain a copy of $\hat{\mathbb{Z}}^2$.*

The proof of theorem 3.10 crucially relies on two important and deep results due to Wise and Zalesskii respecitvely. We begin by recalling a couple of definitions.

Definition 3.11. Let G be a group and let H be a subgroup. We say that H is *malnormal* in G if $gHg^{-1} \cap H = \{1\}$ for every $g \notin H$.

Definition 3.12. Let G be a profinite group acting on a profinite tree T. We say that the action is *k-acylindrical* if for every $g \neq 1$, the set of fixed points of G in T has diameter at most k.

The aforementioned theorems that we will use are the following:

Theorem 3.13. *Every hyperbolic, virtually special group G has a finite-index subgroup that admits a* malnormal quasiconvex hierarchy*: G can be built from trivial groups by repeatedly taking HNN extensions and amalgamated products, such that in each stage the edge groups are quasiconvex and* malnormal *in the amalgamated product or HNN extension.*

The second theorem is a Tits' alternative type theorem for profinite groups acting on profinite trees. The original statement is long and cumbersome, and we therefore state it for profinite groups acting 1-acylindrically on profinite trees (this is the case with splittings coming from the quasiconvex malnormal hierarchy mentioned above):

Theorem 3.14. *Let G be a torsion-free profinite group admitting a 1-acylindrical action on a profinite tree T. Then one of the following holds:*

- *G contains a non-abelian free pro-p subgroup for some p;*
- *G stabilizes a vertex;*

- *$G \cong \prod_{p\in P_1} \mathbb{Z}_p \rtimes \prod_{p\in P_2} \mathbb{Z}_p$ for disjoint sets of primes P_1 and P_2.*

We will also use the following lemma from Wilton's and Zalesskii's paper:

Lemma 3.15. *Let G be a compact virtually special hyperbolic group and let H be a quasiconvex, malnormal subgroup of G. Then the closure $\overline{H}$ of H in $\hat{G}$ is isomorphic to $\hat{H}$ and it is malnormal in $\hat{G}$.*

We are ready to prove Theorem 3.10.

Proof of Theorem 3.10. Let G_0 be a finite-index subgroup of G that admits a malnormal hierarchy, and let H be a subgroup of $\hat{G}$. Let $H_0 = H \cap G_0$ and note that H_0 is a finite-index subgroup of H. Therefore, it is enough to show that H_0 is not isomorphic to $\hat{\mathbb{Z}}^2$.

If H_0 contains a non-abelian free pro-p, then we are done. Otherwise, we will prove that $H_0 \cong \prod_{p\in P_1} \mathbb{Z}_p \rtimes \prod_{p\in P_2} \mathbb{Z}_p$. We do so by induction on the height of the hierarchy. Note that the quasiconvex malnormal splitting of $H_0 \cap G$ lifts to a profinite malnormal splitting of H_0. We first prove that the action of H_0 on the profinite tree T corresponding to this splitting is 1-acylindrical.

Since the splitting of H_0 has a single edge with edge group C, it is enough to show that the stabilizers of any two distinct edges have a trivial intersection. But such an intersection has the form $gCg^{-1} \cap hCh^{-1}$ and must be trivial since C is malnormal in H by the previous lemma. Hence the action of H_0 on T is 1-acylindrical.

Now, recall the Tits' alternative theorem for such actions. We assume that H_0 does not contain a non-abelian free pro-p subgroup, and we are left with two options:

(1) H_0 fixes a vertex, in which case it is contained in a group with a strictly shorter hierarchy, and we are done by the induction hypotheses.
(2) $H_0 \cong \prod_{p\in P_1} \mathbb{Z}_p \rtimes \prod_{p\in P_2} \mathbb{Z}_p$, which completes the proof.

□

References

[Aka12] Menny Aka. Arithmetic groups with isomorphic finite quotients. *Journal of Algebra*, 352(1):322–340, February 2012.

[BG04] Martin R. Bridson and Fritz J. Grunewald. Grothendieck's problems concerning profinite completions and representations of groups. *Annals of Mathematics*, 160(1):359–373, 2004.

[BMRS20] M. A. Bridson, D. B. McReynolds, A. W. Reid, and R. Spitler. Absolute profinite rigidity and hyperbolic geometry. *Annals of Mathematics*, 192(3), November 2020.

[BMRS21] Martin R. Bridson, D. B. McReynolds, Alan W. Reid, and Ryan Spitler. On the profinite rigidity of triangle groups. *Bulletin of the London Mathematical Society*, 53(6):1849–1862, September 2021.

[BR84] G. Baumslag and J. E. Roseblade. Subgroups of direct products of free groups. *Journal of The London Mathematical Society-second Series*, pages 44–52, 1984.

[Bri23] Martin R. Bridson. Profinite isomorphisms and fixed-point properties, 2023.

[BRS23] M. R. Bridson, A. W. Reid, and R. Spitler. Absolute profinite rigidity, direct products, and finite presentability, 2023.

[BW07] Martin R. Bridson and Henry Wilton. Subgroup separability in residually free groups. *Mathematische Zeitschrift*, 260(1):25–30, October 2007.

[CHMV24] Samuel M. Corson, Sam Hughes, Philip Möller, and Olga Varghese. Profinite rigidity of affine coxeter groups, 2024.

[CW22] Tamunonye Cheetham-West. Absolute profinite rigidity of some closed fibered hyperbolic 3-manifolds, 2022.

[CWLRS22] Tamunonye Cheetham-West, Alexander Lubotzky, Alan W. Reid, and Ryan Spitler. Property fa is not a profinite property, 2022.

[DFPR82] John D. Dixon, Edward W. Formanek, John C. Poland, and Luis Ribes. Profinite completions and isomorphic finite quotients. *Journal of Pure and Applied Algebra*, 23(3):227–231, March 1982.

[EK23] Daniel Echtler and Holger Kammeyer. Bounded cohomology is not a profinite invariant. *Canadian Mathematical Bulletin*, 67(2):379–390, October 2023.

[FM24] Jonathan Fruchter and Ismael Morales. Virtual homology of limit groups and profinite rigidity of direct products, 2024.

[Gar19] Giles Gardam. Profinite rigidity in the snappea census. *Experimental Mathematics*, 30(4):489–498, April 2019.

[Ham01] Emily Hamilton. Abelian subgroup separability of haken 3-manifolds and closed hyperbolic n -orbifolds. *Proceedings of the London Mathematical Society*, 83(3):626–646, November 2001.

[JZ23] Andrei Jaikin-Zapirain. The finite and solvable genus of finitely generated free and surface groups. *Research in the Mathematical Sciences*, 10(4), October 2023.

[KS23] Steffen Kionke and Eduard Schesler. Amenability and profinite completions of finitely generated groups. *Groups, Geometry, and Dynamics*, 17(4):1235–1258, August 2023.

[Pic71] P. F. Pickel. Finitely generated nilpotent groups with isomorphic finite quotients. *Transactions of the American Mathematical Society*, 160(0):327–341, 1971.

[PS24] Gianluca Paolini and Rizos Sklinos. Profinite rigidity of affine coxeter groups, 2024.

[PT86] V. P. Platonov and O. I. Tavgen'. On the grothendieck problem of profinite completions of groups. *Dokl. Akad. Nauk SSSR*, 288(5):1054–1058, 1986. MathSciNet: MR852649, Zbl: 0614.20016.

[Wil08] Henry Wilton. Hall's theorem for limit groups. *Geometric and Functional Analysis*, 18(1):271–303, March 2008.

[Wil18] Henry Wilton. Essential surfaces in graph pairs. *Journal of the American Mathematical Society*, 31(4):893–919, June 2018.

[WZ17] Henry Wilton and Pavel Zalesskii. Distinguishing geometries using finite quotients. *Geometry & Topology*, 21(1):345–384, February 2017.